Aequat. Math. 91 (2017), 161–167

0001-9054/17/010161-7
published online January 2, 2017
DOI 10.1007/s00010-016-0455-6

Aequationes Mathematicae

CrossMark

Lipschitzian solutions to linear iterative equations revisited

Karol Baron and Janusz Morawiec

Abstract. We study the problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions φ of equations of the form

$$\varphi(x) = \int_\Omega g(\omega)\varphi\big(f(x,\omega)\big)\mu(d\omega) + F(x),$$

where μ is a measure on a σ-algebra of subsets of Ω and $\int_\Omega g(\omega)\mu(d\omega) = 1$.

Mathematics Subject Classification. Primary 39B12.

Keywords. Iterative equations, Lipschitzian solutions, continuous dependence of solutions, Bochner integral.

1. Introduction

Fix a measure space $(\Omega, \mathcal{A}, \mu)$ and a separable metric space (X, ρ).

We continue a study of Lipschitzian solutions φ of equations of the form

$$\varphi(x) = \int_\Omega g(\omega)\varphi\big(f(x,\omega)\big)\mu(d\omega) + F(x) \tag{1}$$

assuming now, contrary to [2], that

$$\int_\Omega g(\omega)\mu(d\omega) = 1. \tag{2}$$

Concerning the given functions f, g and F we assume the following hypotheses in which $\mathcal{B}$ stands for the σ-algebra of all Borel subsets of X and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(H_1) The function f maps $X \times \Omega$ into X and for every $x \in X$ the function $f(x, \cdot)$ is $\mathcal{A}$-measurable, i.e.

$$\{\omega \in \Omega : f(x,\omega) \in B\} \in \mathcal{A} \quad \text{for all } x \in X \text{ and } B \in \mathcal{B}.$$

(H_2) The function $g\colon \Omega \to \mathbb{K}$ is integrable, satisfies (2),

$$\int_\Omega |g(\omega)|\rho\big(f(x,\omega),x\big)\mu(d\omega) < \infty \quad \text{for every } x \in X,$$

and

$$\int_\Omega |g(\omega)|\rho\big(f(x,\omega),f(z,\omega)\big)\mu(d\omega) \le \lambda\rho(x,z) \qquad \text{for all } x,z \in X$$

with $\lambda \in [0,1)$.

(H_3) The function F maps X into a separable Banach space Y over $\mathbb{K}$ and

$$\|F(x) - F(z)\| \le L\rho(x,z) \qquad \text{for all } x,z \in X$$

with $L \in [0,+\infty)$.

As emphasized in [3, Section 0.3] iteration is the fundamental technique for solving functional equations in a single variable, and iterates usually appear in the formulae for solutions. We iterate, as in [2], the operator which transforms a Lipschitzian $F\colon X \to Y$ into $\int_\Omega g(\omega)F(f(x,\omega))\mu(d\omega)$, contrary to [3, Section 7.2D] where the Schauder fixed point theorem is used. The special case where $g(\omega) = 1$ for all $\omega \in \Omega$ and $\mu(\Omega) = 1$ was considered in [1, Section 4] on a base of iteration of random-valued functions.

For integrating vector functions we use the Bochner integral.

2. Existence and uniqueness

Assuming (H_1)–(H_3) and making use of [2, Lemma 2.2] we define

$$F_0(x) = F(x), \quad F_n(x) = \int_\Omega g(\omega)F_{n-1}(f(x,\omega))\mu(d\omega) \tag{3}$$

for all $x \in X$ and $n \in \mathbb{N}$, and note that

$$\|F_n(x) - F_n(z)\| \le L\lambda^n\rho(x,z) \qquad \text{for all } x,z \in X \text{ and } n \in \mathbb{N}. \tag{4}$$

Our main result reads as follows.

Theorem 2.1. *Assume* (H_1)–(H_3). *Then:*

(i) *There is a* $y_0 \in Y$ *such that* $\lim_{n\to\infty} F_n(x) = y_0$ *for every* $x \in X$;

(ii) *Equation* (1) *has a Lipschitzian solution* $\varphi\colon X \to Y$ *if and only if*

$$\lim_{n\to\infty} F_n(x_0) = 0 \textit{ for an } x_0 \in X.$$

(iii) *Any Lipschitzian solution* $\varphi\colon X \to Y$ *of* (1) *has the form*

$$\varphi(x) = c + \sum_{n=0}^{\infty} F_n(x) \qquad \textit{for every } x \in X, \tag{5}$$

where c *is a constant from* Y.

(iv) *If* $\varphi\colon X \to Y$ *is a Lipschitzian solution of* (1), *then*

$$\|\varphi(x) - \varphi(z)\| \le \frac{L}{1-\lambda}\rho(x,z) \quad \textit{for all } x, z \in X. \tag{6}$$

Proof. It follows from (3), (2) and (4) that

$$\|F_n(x) - F_{n-1}(x)\| \le L\lambda^{n-1} \int_\Omega |g(\omega)|\rho\big(f(x,\omega), x\big)\mu(d\omega)$$

for all $x \in X$ and $n \in \mathbb{N}$. Consequently, for every $x \in X$ the series

$$\sum_{n=1}^{\infty} \big(F_n(x) - F_{n-1}(x)\big)$$

converges, i.e., $(F_n(x))_{n\in\mathbb{N}}$ converges in Y. Hence and from (4) assertion (i) follows.

Passing to the proof of assertion (ii) assume that $\varphi\colon X \to Y$ is a Lipschitzian solution of (1) with a Lipschitz constant L_φ and define

$$\varphi_0(x) = \varphi(x), \quad \varphi_n(x) = \int_\Omega g(\omega)\varphi_{n-1}(f(x,\omega))\mu(d\omega) \tag{7}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since

$$\varphi_{n-1} = \varphi_n + F_{n-1} \quad \text{for every } n \in \mathbb{N},$$

we have

$$\varphi = \varphi_n + \sum_{k=0}^{n-1} F_k \quad \text{for every } n \in \mathbb{N}, \tag{8}$$

and by (i) there is a $c \in Y$ such that

$$\lim_{n\to\infty} \varphi_n(x) = c \quad \text{for every } x \in X.$$

Taking this and (8) into account we see that for every $x \in X$ the series occurring in (5) converges and (5) holds. In particular, $\lim_{n\to\infty} F_n(x) = 0$ for every $x \in X$. Applying (5) and (4) we obtain (6).

For the proof of the existence assume that $\lim_{n\to\infty} F_n(x_0) = 0$ for an $x_0 \in X$ and making use of (4) define $\Phi\colon X \to Y$ by

$$\Phi(x) = \sum_{n=0}^{\infty} \big(F_n(x) - F_n(x_0)\big). \tag{9}$$

Clearly,

$$\|\Phi(x) - \Phi(z)\| \le \frac{L}{1-\lambda}\rho(x,z) \quad \text{for all } x, z \in X. \tag{10}$$

We shall show that Φ solves (1). To this end fix $x \in X$. According to [2, Lemma 2.2] the function

$$\omega \longmapsto g(\omega)\Phi\big(f(x,\omega)\big), \quad \omega \in \Omega,$$

is Bochner integrable and by (4) for all $n \in \mathbb{N}$ and $\omega \in \Omega$ we have

$$\Big\| g(\omega)\Big(F_n\big(f(x,\omega)\big) - F_n(x_0)\Big)\Big\| \le L\lambda^n |g(\omega)|\big(\rho(f(x,\omega),x) + \rho(x,x_0)\big).$$

Hence, applying the dominated convergence theorem, (9), (3) and (2) we see that

$$\begin{aligned}
\int_\Omega g(\omega)\Phi\big(f(x,\omega)\big)\mu(d\omega) &= \sum_{n=0}^{\infty} \int_\Omega g(\omega)\Big(F_n\big(f(x,\omega)\big) - F_n(x_0)\Big)\mu(d\omega) \\
&= \sum_{n=0}^{\infty} \big(F_{n+1}(x) - F_n(x_0)\big) \\
&= \sum_{n=0}^{\infty} \big(F_{n+1}(x) - F_{n+1}(x_0)\big) \\
&\quad + \sum_{n=0}^{\infty} \big(F_{n+1}(x_0) - F_n(x_0)\big) \\
&= \sum_{n=1}^{\infty} \big(F_n(x) - F_n(x_0)\big) \\
&\quad + \lim_{n\to\infty} \big(F_n(x_0) - F_0(x_0)\big) \\
&= \sum_{n=0}^{\infty} \big(F_n(x) - F_n(x_0)\big) - F_0(x) \\
&= \Phi(x) - F(x),
\end{aligned}$$

which ends the proof. □

The following example shows that sometimes the limit of the sequence $(F_n)_{n\in\mathbb{N}}$ can be easily calculated, but its value may be a surprise.

Example 2.2. Given a Lipschitzian $F\colon [0,1] \to \mathbb{R}$ consider the equation

$$\varphi(x) = \frac{1}{2}\varphi\left(\frac{1}{2}x\right) + \frac{1}{2}\varphi\left(\frac{1}{2}x + \frac{1}{2}\right) + F(x). \tag{11}$$

In this case $f(x,\omega) = \frac{1}{2}x + \omega$ and $g(\omega) = 1$ for all $x \in [0,1]$ and $\omega \in \Omega = \{0, \frac{1}{2}\}$, $\mu(\{0\}) = \mu(\{\frac{1}{2}\}) = \frac{1}{2}$, and

$$F_n(x) = \frac{1}{2}F_{n-1}\left(\frac{1}{2}x\right) + \frac{1}{2}F_{n-1}\left(\frac{1}{2}x + \frac{1}{2}\right) = \frac{1}{2^n}\sum_{k=0}^{2^n-1} F\left(\frac{1}{2^n}x + \frac{k}{2^n}\right)$$

for all $x \in [0,1]$ and $n \in \mathbb{N}$, whence

$$\lim_{n\to\infty} F_n(x) = \int_0^1 F(y)dy \quad \text{for every } x \in [0,1].$$

According to Theorem 2.1 Eq. (11) has a Lipschitzian solution $\varphi\colon [0,1] \to \mathbb{R}$ if and only if

$$\int_0^1 F(x)dx = 0.$$

Remark 2.3. Note that [2, Example 3.3] shows that assumptions (H_1)–(H_3) do not guarantee the existence of a *continuous* solution $\varphi\colon X \to Y$ of (1).

Remark 2.4. As [1, Example 4.2] shows, under the assumptions of Theorem 2.1 besides a Lipschitzian solution Eq. (1) may also have a continuous one which is not Lipschitzian.

Remark 2.5. For every $\lambda \in (0,1)$ the logarithmic function restricted to $[1,\infty)$ is a Lipschitzian solution to the equation

$$\varphi(x) = \varphi\big(\lambda x + 1 - \lambda\big) + \log\frac{x}{\lambda x + 1 - \lambda};$$

here $f(x,\omega) = \lambda x + 1 - \lambda$, $g(\omega) = 1$, $F(x) = \log\frac{x}{\lambda x+1-\lambda}$ for all $x \in [1,\infty)$ and $\omega \in \Omega$, $\mu(\Omega) = 1$. According to Theorem 2.1 it is the only up to an additive constant Lipschitzian solution $\varphi\colon [1,\infty) \to \mathbb{R}$ to this equation, and it is unbounded in spite of the fact that F is bounded.

3. Continuous dependence

Assume (H_1) and (H_2), fix $x_0 \in X$ and let $(Y, \|\cdot\|)$ be a separable Banach space over $\mathbb{K}$. In what follows we consider the linear space $Lip(X,Y)$ of all Lipschitzian functions mapping X into Y with the norm

$$\|F\|_{Lip} = \|F(x_0)\| + \|F\|_L,$$

where $\|F\|_L$ stands for the smallest Lipschitz constant for F, and the linear subspace $\mathcal{F}$ of $Lip(X,Y)$ consists of all $F \in Lip(X,Y)$ such that for the sequence $(F_n)_{n\in\mathbb{N}}$ defined by (3) we have $\lim_{n\to\infty} F_n(x_0) = 0$. It is clear that the norm $\|\cdot\|_{Lip}$ depends on the fixed point x_0, but for different points such norms are equivalent, and it follows from (4) that $\mathcal{F}$ does not depend on x_0. Putting

$$\mathcal{F}_0 = \big\{F \in Lip(X,Y) : F(x_0) = 0\big\}$$

we shall prove the following theorem.

Theorem 3.1. *If (H_1), (H_2) hold and Y is a separable Banach space over $\mathbb{K}$, then for every $F \in \mathcal{F}$ the formula*

$$\varphi^F(x) = \sum_{n=0}^{\infty}\big(F_n(x) - F_n(x_0)\big) \quad \textit{for every } x \in X$$

defines the only Lipschitzian and vanishing at x_0 solution $\varphi^F\colon X \to Y$ of (1), the operator

$$F \mapsto \varphi^F, \quad F \in \mathcal{F}, \tag{12}$$

is a linear homeomorphism of $\mathcal{F}$ onto $\mathcal{F}_0$ and

$$\frac{1}{1+\lambda+\int_\Omega |g(\omega)|\rho\big(f(x_0,\omega),x_0\big)\mu(d\omega)}\|F\|_{Lip} \le \|\varphi^F\|_{Lip} \le \frac{1}{1-\lambda}\|F\|_{Lip} \tag{13}$$

for every $F \in \mathcal{F}$.

Proof. In view of Theorem 2.1 we have to show that operator (12) maps $\mathcal{F}$ onto $\mathcal{F}_0$ and for every $F \in \mathcal{F}$ the first inequality in (13) holds.

Fix $\psi \in \mathcal{F}_0$. According to [2, Lemma 2.2] the formula

$$F(x) = \psi(x) - \int_\Omega g(\omega)\psi(f(x,\omega))\mu(d\omega) \quad \text{for every } x \in X$$

defines an $F \in Lip(X,Y)$. Since ψ is a Lipschitzian solution of (1), by Theorem 2.1 the function F is in $\mathcal{F}$ with

$$\|F\|_L \le \|\psi\|_L + \lambda\|\psi\|_L.$$

Moreover, as $\psi(x_0)=0$, $\psi=\varphi^F$ and

$$\begin{aligned}\|F(x_0)\| &\le \int_\Omega |g(\omega)|\|\psi\big(f(x_0,\omega)\big)) - \psi(x_0)\|\mu(d\omega)\\ &\le \|\psi\|_L \int_\Omega |g(\omega)|\rho\big(f(x_0,\omega),x_0\big)\mu(d\omega).\end{aligned}$$

Consequently,

$$\|F\|_{Lip} \le \|\varphi^F\|_{Lip}\left(\int_\Omega |g(\omega)|\rho\big(f(x_0,\omega),x_0\big)\mu(d\omega)+1+\lambda\right),$$

which ends the proof. □

Remark 3.2. Since $\mathcal{F}_0$ is closed in the Banach space $(Lip(X,Y), \|\cdot\|_{Lip})$, it follows from Theorem 3.1 that also $\mathcal{F}$ is a closed subspace of $(Lip(X,Y), \|\cdot\|_{Lip})$.

Acknowledgements

This research was supported by the University of Silesia Mathematics Department (Iterative Functional Equations and Real Analysis program).

References

[1] Baron, K.: On the convergence in law of iterates of random-valued functions. Aust. J. Math. Anal. Appl. **6**(1), 9 (2009) (**Art. 3**)
[2] Baron, K., Morawiec, J.: Lipschitzian solutions to linear iterative equations. Publ. Math. Debr. **89**(3), 277–285 (2016)
[3] Kuczma, M., Choczewski, B., Ger, R.: Iterative functional equations, Encyclopedia of Mathematics and its Applications, vol. 32. Cambridge University Press, Cambridge (1990)

Karol Baron and Janusz Morawiec
Instytut Matematyki
Uniwersytet Śląski
Bankowa 14
40-007 Katowice
Poland
e-mail: baron@us.edu.pl

Janusz Morawiec
e-mail: morawiec@math.us.edu.pl

Received: July 14, 2016